Function Spaces
End Semester Exam
Marks - 50, Duration - 3 Hours

1. [3 marks] Let M and N be subsets of $\mathbb{R}^n$. Define

$$M + N = \{x + y : x \in M, y \in N\}.$$

Pick out the true statements:

(a) If M and N are closed subspaces, then $M + N$ is a closed subspace.
(b) If M is an open set and if N is a set, then $M + N$ is an open set.
(c) If M and N are compact sets, then $M + N$ is a compact set.

2. [3 marks] Which of the following statements are true?

(a) Let A and B be two subsets of a metric space (M, d). If $d(A, B) > 0$, then there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \phi$.
(b) Let $f : (0, \infty) \to (0, \infty)$ be such that $|f(x) - f(y)| \leq \frac{1}{2}|x - y|$ for all $x, y > 0$. Then f has a fixed point.
(c) Let φ, ψ be continuous functions on $[0, 1]$. Let $\{f_n\}_{n\geq 1}$ be a sequence in $(C[0, 1], \|.\|_\infty)$ such that, for all n, the functions f_n are continuously differentiable and $|f_n(x)| \leq \varphi(x)$ and $|f_n'(x)| \leq \psi(x)$ for all $x \in [0, 1]$ and for all n. Then there exists a subsequence of $\{f_n\}_{n\geq 1}$ which converges in $(C[0, 1], \|.\|_\infty)$.

3. [3 marks] Let $f \in C^1[-\pi, \pi]$, the space of real-valued continuously differentiable functions on $[-\pi, \pi]$, be such that $f(-\pi) = f(\pi)$. Define

$$a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos ntdt \quad (n \geq 1).$$

Which of the following statements are true?

(a) The sequence $\{a_n\}_{n\geq 1}$ is bounded.
(b) The series $\sum_{n=1}^{\infty} n^2 |a_n|^2$ is convergent.
(c) The series $\sum_{n=1}^{\infty} |a_n|$ is convergent

4. [3 marks] Let $f : [-\pi, \pi] \to \mathbb{C}$ be a continuous 2π-periodic function whose Fourier series is given by

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \quad (a_k, b_k \in \mathbb{C}).$$

Let for each n, the partial sum of the Fourier series is

$$S_n(x) := a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

and let S_0 denote the constant function $\frac{a_0}{2}$. Which of the following statements are true?

(a) $s_n \to f$ uniformly on $[-\pi, \pi]$.
(b) If $\sigma_n = \frac{s_0+s_1\cdots+s_n}{n+1}$, then $\sigma_n \to f$ uniformly on $[-\pi, \pi]$.
(c) $\int_{-\pi}^{\pi} |s_n(t) - f(t)|^2 dt \to 0$, as $n \to \infty$.

5. [3 marks] Let $f : [-\pi, \pi] \to \mathbb{C}$ be continuous. Pick out the case(s) which imply that $f = 0$.
   (a) $f(-t) = f(t)$ for all $t \in [0, \pi]$ and
   $$\int_{-\pi}^{\pi} f(t) \cos ntdt = 0 \quad (n \geq 0).$$
   (b) $f(-t) = -f(t)$ for all $t \in [0, \pi]$, and
   $$\int_{-\pi}^{\pi} f(t) \sin ntdt = 0 \quad (n \geq 1).$$
   (c)
   $$\int_{-\pi}^{\pi} f(t) \cos ntdt = 0 \quad (n \geq 0) \quad \text{and} \quad \int_{-\pi}^{\pi} f(t) \sin ntdt = 0 \quad (n \geq 1).$$

6. [5 marks] Suppose f is a real-valued continuous function on $(\mathbb{R}^n, \|.\|_2)$ with the property that there is a number $c > 0$ such that $|f(x)| \geq c\|x\|_2$ for all $x \in \mathbb{R}$. Show that if K is a compact subset of $\mathbb{R}$, then $f^{-1}(K)$ is compact set in $\mathbb{R}^n$.

7. [6 marks] Let (M, d) be a complete metric space. Assume that each $f_n : M \to M$ has at least a fixed point $x_n \in M$, that is, $x_n = f_n(x_n)$. Assume that $f : M \to M$ is a contraction. Show that if f_n converges uniformly to f, then x_n converges to the fixed point f.

8. [3+4 marks] Let $K(x, t)$ be a continuous function on the square $[a, b] \times [a, b]$.
   (a) Given $f \in C[a, b]$, show that
   $$g(x) = \int_a^b f(t)K(x, t)\, dt$$
   defines a continuous function in $C[a, b]$.
   (b) Define $T : C[a, b] \to C[a, b]$ by
   $$(Tf)(x) = \int_a^b f(t)K(x, t)\, dt.$$
   Show that T maps bounded sets into equicontinuous sets. In particular, T is continuous.

9. [7 marks] Prove that differentiability of f at a point implies convergence of its Fourier series at the point.

10. [2+5+3 marks] Let $f : [0, 2\pi] \to \mathbb{R}$ be a function defined by
   $$f(x) = (\pi - x)^2 \quad (x \in [0, 2\pi]).$$
   (a) First extend f on $\mathbb{R}$ as a 2π-periodic continuous function and twice differentiable function on $\mathbb{R}$.
   (b) Show that the Fourier series of f is
   $$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$
   Is the above series convergent uniformly on $\mathbb{R}$? If it is true, then what is the limit?
   (c) Prove that
   $$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$